Algebraic Differential Characters of Flat Connections with Nilpotent Residues

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Abstract We construct unramified algebraic differential characters for flat connections with nilpotent residues along a strict normal crossings divisor.

1 Introduction

In [1], Chern and Simons defined classes $\hat{c}_n((E,\nabla)) \in H^{2n-1}(X, \mathbb{R}/\mathbb{Z}(n))$ for $n \geq 1$ and a flat bundle (E,∇) on a $\mathcal{C}^\infty$ manifold X, where $\mathbb{Z}(n) := \mathbb{Z}\cdot(2\pi\sqrt{-1})^n$. Cheeger and Simons defined in [2] the group of real $\mathcal{C}^\infty$ differential characters $\hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$, which is an extension of global $\mathbb{R}$-valued $2n$-closed forms with $\mathbb{Z}(n)$-periods by $H^{2n-1}(X, \mathbb{R}/\mathbb{Z}(n))$. They show that the Chern–Simons classes extend to classes $\hat{c}_n((E,\nabla)) \in \hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$, if ∇ is a (not necessarily flat) connection, such that the associated differential form is the Chern form computing the nth Chern class associated to the curvature of ∇.

If X now is a complex manifold, and (E,∇) is a bundle with an algebraic connection, Chern–Simons and Cheeger–Simons invariants give classes $\hat{c}_n((E,\nabla)) \in \hat{H}^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$ with a similar definition of complex $\mathcal{C}^\infty$ differential characters. Those classes have been studied by various authors, and most remarquably, it was shown by Reznikov that if X is projective and (E,∇) is flat, then the classes $\hat{c}_n((E,\nabla))$ are torsion, for $n \geq 2$. This answered positively a conjecture by Bloch [3], which echoed a similar conjecture by Cheeger–Simons in the $\mathcal{C}^\infty$ category [2, 4].

On the other hand, for X a smooth complex algebraic variety, we defined in [5] the group $AD^n(X)$ of algebraic differential characters. It is easily written as the hypercohomology group $\mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X \xrightarrow{d} \Omega^{n+1}_X \to \dots \xrightarrow{d} \Omega^{2n-1}_X)$, where $\mathcal{K}_n$ is the Zariski sheaf of Milnor K-theory which is unramified in codimension 1.

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It has the property that it maps to the Chow group $CH^n(X)$, to algebraic closed $2n$-forms which have $\mathbb{Z}(n)$-periods, and to the complex $\mathcal{C}^\infty$ differential characters $\hat{H}^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$. If (E, ∇) is a bundle with an algebraic connection, it has classes $c_n((E, \nabla)) \in AD^n(X)$ which lift both the Chern classes of E in $CH^n(X)$ and $\hat{c}_n((E, \nabla))$. All those constructions are contravariant in $(X, (E, \nabla))$, the differential characters have an algebra structure, and the classes fulfill the Whitney product formula. They admit a logarithmic version: if $j : U \to X$ is a (partial) smooth compactification of U such that $D := X \setminus U$ is a strict normal crossings divisor, one defines the group $AD^n(X, D) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_X(\log D) \xrightarrow{d} \Omega^{n+1}_X(\log D) \to \ldots \xrightarrow{d} \Omega^{2n-1}_X(\log D))$. Obviously one has maps $AD^n(X) \to AD^i(X, D) \to AD^n(U)$. The point is that if (E, ∇) extends a pole free connection $(E, \nabla)|_U$ to a connection on X with logarithmic poles along D, then $c_n((E, \nabla)|_U) \in AD^n(U)$ lifts to well defined classes $c_n((E, \nabla)) \in AD^n(X, D)$ with the same functoriality and additivity properties.

If X is a smooth algebraic variety defined over a characteristic 0 field, and $X \supset U$ is a smooth (partial) compactification of U, it is computed in [6, Appendix B] that one can express the Atiyah class [7] of a bundle extension E of $E|_U$ in terms the residues of the extension ∇ of $\nabla|_U$ along $D = X \setminus U$. In particular, if X is projective, ∇ has logarithmic poles along D and has nilpotent residues, one obtains that the de Rham Chern classes of E are zero. If $k = \mathbb{C}$, this implies that the (analytic) Chern classes of E in Deligne–Beilinson cohomology $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ lie in the continuous part $H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))/F^n \subset H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$.

The purpose of this note is to show that this lifting property is in fact stronger.

Theorem 1. *Let $X \supset U$ be a smooth (partial) compactification of a complex variety U, such that $D = \sum_j D_j = X \setminus U$ is a strict normal crossings divisor. Let (E, ∇) be a flat connection with logarithmic poles along D such that its residues Γ_j along D_j are all nilpotent. Then the classes $c_n((E, \nabla)) \in AD^n(X, D)$ lift to well defined classes $c_n((E, \nabla, \Gamma)) \in AD^n(X)$, which satisfy the Whitney product formula. More precisely, the classes $c_n((E, \nabla, \Gamma))$ lie in the subgroup $AD^n_\infty(X) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_{\bar{X}} \xrightarrow{d} \Omega^{n+1}_X \to \ldots \xrightarrow{d} \Omega^{\dim(X)}_X) \subset AD^n(X)$ of classes mapping to 0 in $H^0(X, \Omega^{2n}_X)$.*

They also fulfill some functoriality property, and one can express what their restriction to the various strata of D precisely are.

Let us denote by $\hat{c}_n((E, \nabla, \Gamma))$ the image of $c_n((E, \nabla, \Gamma))$ via the regulator map $AD^n(X) \to \hat{H}^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$ defined in [5] and [8], which restricts to a regulator map $AD^n_\infty(X) \to H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$. As an immediate consequence, one obtains the following:

Corollary 2. *Let $(X, (E, \nabla, \Gamma))$ be as in the theorem. Then the Cheeger–Chern–Simons classes $\hat{c}_n((E, \nabla)|_U) \in H^{2n-1}(U_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(U_{\text{an}}, \mathbb{C}/\mathbb{Z})$ lift to well defined classes $\hat{c}_n((E, \nabla, \Gamma)) \in H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$, with the same properties.*

A direct $\mathcal{C}^\infty$ construction of $\hat{c}_n((E, \nabla, \Gamma)) \in H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$ in the spirit of Cheeger–Chern–Simons has been performed by Deligne and is written in a letter of Deligne to the authors of [9]. It consists in modifying the given connection ∇ by a $\mathcal{C}^\infty$ one form with values in $\mathcal{E}nd(E)$, so as to obtain a (possibly non-flat) connection without residues along D. This modified connection admits classes in $H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \subset \hat{H}^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z})$. That they do not depend on the choice of the one form relies essentially on the argument showing that if ∇ is flat with logarithmic poles along D (and without further conditions on the residues), for $n \geq 2$, the image of $c_n((E, \nabla))$ in $H^0(U, \mathcal{H}^{2n-1}_{DR})$, where $\mathcal{H}^j_{DR}$ is the Zariksi sheaf of j-th de Rham cohomology, in fact lies in the unramified cohomology $H^0(X, \mathcal{H}^{2n-1}_{DR}) \subset H^0(U, \mathcal{H}^{2n-1}_{DR})$. For this, see [10, Theorem 6.1.1]. In the case when D is smooth, Iyer and Simpson constructed the $\mathcal{C}^\infty$ classes $\hat{c}_n((E, \nabla, \Gamma)) \in H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$ using the existence of the $\mathcal{C}^\infty$ trivialization of the canonical extension after an étale cover, a fact written by Deligne in a letter, together with Deligne's suggestion of considering patched connection. They then show that Reznikov's argument and theorem [11] adapts to those classes. Our note is motivated by the question raised in [9] on the construction in the general case.

Our algebraic construction in Theorem 1 relies on the modified splitting principle developed in [5, 8, 12] in order to define the classes in $AD^n(X, D)$. Let $q : Q \to X$ be the complete flag bundle of E. A flat connection on E with logarithmic poles along D defines a map of differential graded algebras $\tau : \Omega^\bullet_Q(\log q^{-1}(D)) \to \mathcal{K}^\bullet$ where $\mathcal{K}^i = q^*\Omega^i_X(\log D)$ and $Rq_*\mathcal{K}^\bullet = \Omega^\bullet_X(\log D)$. This defines a partial flat connection $\tau \circ q^*\nabla : q^*E \to q^*\Omega^1_X(\log D) \otimes_{\mathcal{O}_Q} q^*E$ which has the property that it stabilizes all the rank one subquotients of q^*E. On the other hand, the nilpotency of Γ allows to filter the restriction $E|_\Sigma$ to the different strata Σ of D, in such a way that the restriction $\nabla|_\Sigma : E|_\Sigma \to \Omega^1_X(\log D)|_\Sigma \otimes E|_\Sigma$ of the connection stabilizes the filtration $F^\bullet_\Sigma$, and has the following important extra property: the induced flat connection $\nabla|_\Sigma$ on $gr(F^\bullet_\Sigma)$ has values in $\Omega^1_\Sigma(\log \text{rest})$, where rest is the interestion with Σ of the part of D which is transversal to Σ. This fact translates into a sort of stratification of the flag bundle Q, where τ is refined on this stratification and has values in the pull back of $\Omega^1_\Sigma(\text{rest})$. Modulo some geometry in Q, the next observation consists in expressing the sections $\alpha \in \Omega^i_X$ of forms without poles as pairs $\alpha = (\beta \oplus \gamma) \in \Omega^1_X(\log D) \oplus \Omega^1_D$ such that $\beta|_D = \gamma$, where $\Omega^i_D = \Omega^i_X / \Omega^i_X(\log D)(-D) \subset \Omega^i_X(\log D)|_D$. This yields a complex receiving quasi-isomorphically $\Omega^{\geq i}_X$, which is convenient to define the wished classes.

2 Filtrations

Let X be a smooth variety defined over a characteristic 0 field k. Let $D \subset X$ be a strict normal crossings divisor (i.e., the irreducible components are smooth over k), and let (E, ∇) be a connection $\nabla : E \to \Omega^1_X(\log D) \otimes E$ with residue Γ defined by the composition

$$\begin{array}{ccc} E & \xrightarrow{\nabla} & \Omega^1_X(\log D) \otimes_{\mathcal{O}_X} E \\ & \searrow^{\Gamma} & \downarrow{\scriptstyle 1\otimes \mathrm{res}} \\ & & \nu_* \mathcal{O}_{D^{(1)}} \otimes_{\mathcal{O}_X} E \end{array} \tag{1}$$

where $D^{(1)} = \sqcup_j D_j$. The composition of Γ with the projection $\nu_*\mathcal{O}_{D^{(1)}} \to \mathcal{O}_{D_j}$ defines $\Gamma_j : E \to \mathcal{O}_{D_j} \otimes E$ which factors through $\Gamma_j \in \mathrm{End}(\mathcal{O}_{D_j} \otimes E)$. We write

$$\Gamma \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_D \otimes_{\mathcal{O}_X} E, \nu_*\mathcal{O}_{D^{(1)}} \otimes_{\mathcal{O}_X} E). \tag{2}$$

Recall that if ∇ is integrable, then

$$[\Gamma_i|_{D_{ij}}, \Gamma_j|_{D_{ij}}] = 0. \tag{3}$$

We use the notation $D_I = D_{i_1} \cap \ldots \cap D_{i_r}$ if $I = \{i_1, \ldots, i_r\}$, $D = D^I + \sum_{s\in I} D_s$ with $D^I = \sum_{\ell \notin I} D_\ell$. The connection $\nabla : E \to \Omega^1_X(\log D) \otimes E$ stabilizes $E(-D_j)$, but also $E \otimes \mathcal{I}_{D_I}$, as the Kähler differential on $\mathcal{O}_X$ restricts to a flat $\Omega^1_X(\log(\sum_{s\in I} D_s))$-connection on $\mathcal{I}_{D_I}$. Thus ∇ induces a flat connection

$$\nabla_I : E|_{D_I} \to \Omega^1_X(\log D)|_{D_I} \otimes E|_{D_I}. \tag{4}$$

One has the diagram

$$\begin{array}{ccccc} & & \Omega^1_{D_j}(\log(D^j \cap D_j)) \otimes E & & \Omega^2_{D_j}(\log(D^j \cap D_j)) \otimes E \\ & & \downarrow & & \downarrow \\ E|_{D_j} & \xrightarrow{\nabla_j} & \Omega^1_X(\log D)|_{D_j} \otimes E & \xrightarrow{\nabla_j} & \Omega^2_X(\log D)|_{D_j} \otimes E \\ & \searrow^{\Gamma_j} & \downarrow{\scriptstyle \mathrm{res}} & \searrow^{1\otimes \mathrm{j}} & \downarrow{\scriptstyle \mathrm{res}} \\ & & E|_{D_j} & & \Omega^1_{D_j}(\log(D^j \cap D_j)) \otimes E \end{array} \tag{5}$$

We define $F^1_j = \mathrm{Ker}(\Gamma_j) \subset E|_{D_j}$. It is a coherent subsheaf. ∇_j sends F^1_j to $\Omega^1_{D_j}(\log D^j \cap D_j) \otimes E$, but because of integrability, the diagram (3) shows that ∇_{D_j} induces a flat connection $F^1_j \to \Omega^1_{D_j}(\log D^j \cap D_j) \otimes F^1_j$.

Claim 1. $F^1_j \subset E|_{D_j}$ is a subbundle.

Proof. We use Deligne's Riemann–Hilbert correspondence [13]: the data are defined over a field of finite type k_0 over $\mathbb{Q}$, so embeddable in $\mathbb{C}$, and the question is compatible with the base changes $\otimes_{k_0} k$ and $\otimes_k \mathbb{C}$. So it is enough to consider the question for the underlying analytic connection on a polydisk $(\Delta^*)^r \times \Delta^s$ with coordinates x_j, where D_j is defined by $x_j = 0$ for $1 \le j \le r$. By the Riemann–Hilbert correspondence, the argument given in [13, p. 86] shows that the analytic connection is isomorphic to $(V \otimes \mathcal{O}, \sum_1^r \Gamma_j^0 \frac{dx_i}{x_i})$, where the matrices Γ_j^0 are constant nilpotent. Thus F_j^1 is isomorphic to $F_j^1(V) \otimes \mathcal{O}_{D_j}$ on the polydisk, with $F_j^1(V) := \mathrm{Ker}(\Gamma_j^0)$, thus is a subbundle. □

We can replace $E|_{D_j}$ by $E|_{D_j}/F_j^1$ in 4 and redo the construction. This defines by pull back $F_j^2 \subset E|_{D_j} \twoheadrightarrow \mathrm{Ker}(\Gamma_j : E|_{D_j}/F_j^1 \to E|_{D_j}/F_j^1)$ with $F_j^2 \supset F_j^1$ etc.

Claim 2. $F_j^\bullet : F_j^0 = 0 \subset F_j^1 \subset \ldots \subset F_j^i \subset \ldots \subset F_j^{r_j} = E|_{D_j}$ is a filtration by subbundles with a flat $\Omega_X^1(\log D)|_{D_j}$-valued connection, such that the induced connection ∇_j on $gr(F_j^\bullet)$ is flat and $\Omega_{D_j}^1(\log D^j \cap D_j)$-valued. (One can also tautologically say that $F_j^\bullet$ refines the (trivial) filtration on $E|_{D_j}$).

Proof. By construction, the flat $\Omega_X^1(\log D)|_{D_j}$-valued connection ∇_j on $E|_{D_j}$ respects the filtration and induces a flat $\Omega_{D_j}^1(\log D^j \cap D_j)$-connection on $gr(F_j^\bullet)$. We use the transcendental argument to show that this is a filtration by subbundles. With the notations as in the proof of the Claim 1, F_j^s is analytically isomorphic to $F_j^s(V) \otimes \mathcal{O}_{D_j}$, where $F_j^1(V) \subset F_j^2(V) \subset \ldots \subset V$ is the filtration on V defined by the successive kernels of Γ_j^0, so $F_j^2(V)$ is the inverse image of $\mathrm{Ker}(\Gamma_j^0)$ on $V/F_j^1(V)$, etc. □

The argument which allows us to construct $F_j^\bullet$ can in be used to define successive refinements on all $E|_{D_I}$. We consider now the case $|I| = r \geq 2$. We refine the filtrations $F_J^\bullet|_{D_I}$, which have been constructed inductively, where $J \subset I, |J| < r$. In fact, we do the construction directly on $E|_{D_I}$. We have r linear maps induced by Γ_j

$$\Gamma_j|_{D_I} : E|_{D_I} \xrightarrow{\nabla_I} \Omega_X^1(\log(D))|_{D_I} \otimes E|_{D_I} \to \mathcal{O}_{D_j} \otimes E|_{D_I} = E_{D_I} \qquad (6)$$

We define

$$F_I^1 = \cap_{j \in I} \mathrm{Ker}(\Gamma_j|_{D_I}) = \cap_{j \in I} F_j^1|_{D_I}. \qquad (7)$$

Claim 3. $F_I^1 \subset E|_{D_I}$ is a subbundle, stabilized by the connection ∇_I, and more precisely one has $\nabla_I : F_I^1 \to \Omega_{D_I}^1(\log(D^I \cap D_I)) \otimes F_I^1$.

Proof. We argue analytically as in the proof of Claim 1. With notations as there, the analytic F_I^1 isomorphic to $F_I^1(V) \otimes \mathcal{O}_{D_I}$. □

Thus ∇_I induces a flat $\Omega_X^1(\log D)|_{D_I}$-valued connection on the quotient $E|_{D_I}/F_I^1$. We define $F_I^2 \supset F_I^1$ in $E|_{D_I}$ to be the inverse image via the projection $E|_{D_I} \to E|_{D_I}/F_I^1$ of $\cap_{j \in I} \mathrm{Ker}(\Gamma_j|_{D_I})$, etc.

Claim 4. The filtration $F_I^\bullet : F_I^0 = 0 \subset F_I^1 \subset F_I^2 \subset \ldots \subset F_I^{r_I} = E|_{D_I}$ is a filtration by subbundles, stabilized by ∇_I, such that ∇_I on $gr(F_I^\bullet)$ is a flat $\Omega_{D_I}^1(\log(D^I \cap D_I))$-valued connection. Furthermore, $F_I^\bullet$ refines all $F_J^\bullet|_{D_I}$ for all $J \subset I, |J| < r$ and one has compatibility of the refinements in the sense that if

$K \subset J \subset I$, then the refinement $F_I^\bullet$ of $F_K^\bullet|_{D_I}$ is the composition of the refinements $F_I^\bullet$ of $F_J^\bullet|_{D_J}$ and $F_J^\bullet$ of $F_K^\bullet|_{D_J}$.

Proof. We argue again analytically. Then F_I^s is isomorphic to $F_I^s(V) \otimes \mathcal{O}_{D_I}$ with the same definition. The filtration terminates as finitely many mutually commuting nilpotent endomorphisms on a finite dimensional vector space always have a common eigenvector. □

Definition 3. We call $F_I^\bullet$ the canonical filtration of $E|_{D_I}$ associated to ∇, which defines $(gr(F_I^\bullet), \nabla_I, \Gamma_I)$ where ∇_I is the flat $\Omega_I^1(\log(D^I \cap D_I)$-valued connection on $gr(F_I^\bullet)$, and Γ_I is its nilpotent residue along the normalization of $D^I \cap D_I$.

Proof.

Au: Please check if "Proof" could be deleted or provide appropriate text.

3 τ-Splittings

We first define flag bundles. We set $q_I : Q_I \to D_I$ to be the total flag bundle associated to $E|_{D_I}$. So the pull back of $E|_{D_I}$ to Q_I has a filtration by subbundles such that the associated graded bundle is a sum of rank one bundles ξ_I^s for $s = 1, \ldots, N = \text{rank}(E)$ (It is here understood that $D_\emptyset = X$, and to simplify, we set $q = q_\emptyset : Q \to X, Q_\emptyset = Q$). For $J \subset I$, the inclusion $D_I \to D_I$ defines inclusions $i(J \subset I) : Q_I \to Q_J$. The canonical filtrations associated to ∇ allow to define partial sections of the q_I. As an illustration, let us assume that $I = \{1\}$, thus D is smooth, and that $F_1^\bullet$ is a total flag, i.e., the $gr(F_1^\bullet)$ is a sum of rank one bundles. Then $F_1^\bullet$ defines a section $D \xrightarrow{\lambda_1^F} Q$.

More generally, let us define $G_I^s = F_I^s / F_I^{s-1}$. We define

$$\begin{array}{ccc} Q_I & \xleftarrow{\lambda_I^F} & Q_I^F \\ {\scriptstyle q_I}\downarrow & \swarrow{\scriptstyle q_I^F} & \\ D_I & & \end{array} \tag{1}$$

using the filtration: recall that $Q_I \to D_I$ is the composition of $\mathbb{P}(E|_{D_I}) \to D_I$ with $\mathbb{P}(E') \to \mathbb{P}(E|_{D_I})$ etc., where $E' \to \mathcal{O}_{\mathbb{P}(E)} \otimes E$ is the rank $(N-1)$ subbundle defined as the kernel to the rank 1 canonical rank 1 bundle $\xi_I^N(\mathbb{P}(E|_{D_I}))$, the pull back of which to Q_I defines the last graded rank 1 quotient. Then the quotient $E|_{D_I} \to G_I^{r_I}$ defines a map $\mathbb{P}(E|_{D_I}) \leftarrow \mathbb{P}(G_I^{r_I})$ such that the pull back of $\xi_I^N(\mathbb{P}(E|_{D_I}))$ is ξ, where ξ is the canonical rank 1 bundle. Writing $G' \to G_I^{r_I}$ for the kernel, we redo the same construction for E', G' replacing $E|_{D_I}, G_I^{r_I}$ etc. We find this way that the flag bundle of $G_I^{r_I}$ maps to the intermediate step between D_I and Q_I which splits the first M rank 1 bundles, where M is the rank of $G_I^{r_I}$. Then we continue with the pull back of $G_I^{r_I-1}$ to the flag bundle of $G_I^{r_I}$, replacing $G_I^{r_I}$, and E'' replacing E, where E'' on this intermediate step is the rank $N - M$ bundle which is not yet split. All this is very classical.

We have extra closed embeddings $\lambda^F(I \subset J)$ which come from the refinements of the canonical filtrations, which are described in the same way: for $J \subset I$, one has commutative squares

$$\begin{array}{ccccccc} Q_J^F & \xleftarrow{\lambda^F(I\subset J)} & Q_I^F & & Q & \xleftarrow{\mu_I} & Q_I^F \\ \downarrow{\scriptstyle q_J^F} & & \downarrow{\scriptstyle q_I^F} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q_I^F} \\ D_J & \xleftarrow[i(I\subset J)]{} & D_I & & X & \xleftarrow[i_I]{} & D_I \end{array} \tag{2}$$

where $i_I = i(\emptyset \subset I)$, $\mu_I = \lambda(\emptyset \subset I)$.

Recall from [5, 8, 12] that ∇ yields a splitting $\tau : \Omega^1_Q(\log q^{-1}(D)) \to q^*\Omega^1_X\,(\log D)$, and that flatness of ∇ implies flatness of τ in the sense that it induces a map of differential graded algebras $(\Omega^\bullet_Q(\log q^{-1}(D)), d) \to (q^*\Omega^\bullet_X(\log D), d_\tau)$ so in particular, $(Rq_*\Omega_X^{\geq n}(\log D), d) = (\Omega_X^{\geq n}(\log D), d)$. Furthermore, the filtration on $q^*(E)$ which defines the rank one subquotient ξ^s has the property that it is stabilized by $\tau \circ q^*\nabla$, and this defines a τ-flat connection $\xi^s \to q^*\Omega^1_X(\log D) \otimes \xi^s$.

The τ-splitting is constructed first on $\mathbb{P}(E)$, with $p : \mathbb{P}(E) \to X$. Then $\tau \circ \nabla$ stabilizes the beginning of the flag $E' \subset$pull-back of E etc. Concretely, the composition $\Omega^1_{\mathbb{P}(E)/X}(1) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}(E)} \otimes E \xrightarrow{\text{projection}} \Omega^1_{\mathbb{P}(E)} \otimes \mathcal{O}_{\mathbb{P}(E)}(1)$ defines the splitting. On the other hand, the flat $\Omega^1_X(\log D)|_{D_I}$-valued connection on $G_I^{r_I}$ has values in $\Omega^1_{D_I}(\log(D_I \cap D^I))$.

When we restrict to $\mathbb{P}(G_I^{r_I})$, then one has a factorization

$$\begin{array}{ccc} \Omega^1_{\mathbb{P}(E)}(\log p^{-1}(D)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} & \xrightarrow{\tau(G_I^{r_I})} & \Omega^1_{D_I}(\log(D_I \cap D^I)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} \\ & \searrow^{\tau} & \downarrow{\scriptstyle \text{inj}} \\ & & \Omega^1_X(\log D) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})} \end{array} \tag{3}$$

which defines a differential graded algebra $(\Omega^\bullet_{D_I}(\log(D_I \cap D^I)) \otimes \mathcal{O}_{\mathbb{P}(G_I^{r_I})}, d_\tau)$ with total direct image on D_I being $(\Omega^\bullet_{D_I}(\log(D_I \cap D^I)), d)$ and with the property that ξ has a flat connection with values in $\Omega^1_{D_I}(\log(D_I \cap D^I))$, which is compatible with the flat $p^*\Omega^1_X(\log D)$-connection on ξ^N. We can repeat the construction with $D_I \to X$ replaced by $\mathbb{P}(G_I^{r_I}) \to \mathbb{P}(E|_{D_I})$, with $E|_{D_I} \to G_I^{r_I}$ replaced by $E' \to G'$ where $E' = \mathrm{Ker}(E|_{D_I} \otimes \mathcal{O}_{\mathbb{P}(E|_{D_I})} \to \mathcal{O}(1))$ and $G' = \mathrm{Ker}(G_I^{r_I} \to \mathcal{O}(1)$. This splits the next rank 1 piece, 1 still has the splitting as in (3), and we go on till we reach the total flag bundle to $G_I^{r_I}$. Then we continue with the flag bundle to $G_I^{r_I - 1}$ etc. We conclude

Claim 5. One has a factorization

$$\begin{array}{ccc} \mu_I^*\Omega^1_Q(\log q^{-1}(D)) & \xrightarrow{\tau_I} & (q_I^F)^*\Omega^1_{D_I}(\log(D_I \cap D^I)) \\ & \searrow^{\tau} & \downarrow \text{inj} \\ & & (q_I^F)^*\Omega^1_X(\log D)|_{D_I} \end{array} \tag{4}$$

τ_I defines a differential graded algebra $((q_I^F)^*\Omega^\bullet_{D_I}(\log(D_I \cap D^I)), d_\tau)$ which is a quotient of $\mu_I^*(\Omega^\bullet_Q(\log q^{-1}(D)), d)$. The flat $q^*\Omega^1_X(\log D)$-valued τ-connection on $\xi^s, s = 1, \ldots, N$, restricts via the splitting τ_I, to a flat $(q_I^F)^*\Omega^1_{D_I}(\log(D^I \cap D_I))$-valued τ-connection on $(\xi_I^F)^s = \mu_I^*\xi^s$.

Definition 4. On Q we define the complex of sheaves

$$A(n) = A^n \to A^{n+1} \to \ldots$$

with

$$\begin{aligned} A^i &= B^i \oplus C^i \\ B^i &= \oplus_I (\mu_I)_*(q_I^F)^*\Omega^i_{D_I}(\log(D^I \cap D_I)), \\ C^i &= \oplus_{I \neq \emptyset}(\mu_I)_*(q_I^F)^*\Omega^{i-1}_X(\log D)|_{D_I}, \end{aligned}$$

where $C^i = 0$ for $i = n$. The differentials D_τ are defined as follows: $(\oplus_I \beta_I, \oplus_I \gamma_I)$, where $\beta_I \in (\mu_I)_*(q_I^F)^*\Omega^i_{D_I}(\log(D^I \cap D_I)), \gamma_I \in (\mu_I)_*(q_I^F)^*\Omega^{i-1}_X(\log D)|_{D_I}$ is sent to

$$\begin{aligned} &\oplus_I d_\tau \beta_I \in (\mu_I)_*(q_I^F)^*\Omega^{i+1}_{D_I}(\log(D^I \cap D_I)), \\ &\qquad \oplus_I d_\tau \gamma_I + (-1)^i(\mu_I^*\beta - \beta_I) \in (\mu_I)_*(q_I^F)^*\Omega^i_X(\log D)|_{D_I}. \end{aligned}$$

Let $\mathcal{K}_n$ be the image of the Zariski sheaf of Milnor K-theory into Milnor K-theory $K_n(k(X)$ of the function field (which is the same as $\text{Ker}(K_n(k(X)) \to \oplus K_{n-1}(\kappa(x)))$ on all codimension 1 points $x \in X$). The τ-differential defines $d_\tau \log : \mathcal{K}_n \to A^n = B^n$ $(C^n = 0)$. The image in A^n is D_τ-flat. Thus this defines $d_\tau \log : \mathcal{K}_n \to A(n)[-1]$.

Definition 5. We define $\mathcal{K}_n\Omega^\infty_Q$ to be the complex $\mathcal{K}_n \xrightarrow{d_\tau \log} A(n)[-1]$ and $\mathcal{K}_n\Omega^\infty_Q \supset (\mathcal{K}_n\Omega^\infty_Q)_0$ to be the subcomplex $\mathcal{K}_n \xrightarrow{d_\tau \log} A^n_{D_\tau}$, where $A^n_{D_\tau}$ means the subsheaf of D_τ-closed sections.

Lemma 6. *The τ-connections on $(\xi_I^F)^s$ define a class $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega^\infty_Q)_0)$ with the property that the image of $\xi^s(\nabla)$ in $H^1(Q, \mathcal{K}_1)$ is $c_1(\xi^s)$.*

Proof. The cocycle of the class $\xi^s(\nabla)$ results from the Claim 5. Write $g^s_{\alpha\beta}$ for a $\mathcal{K}_1$-valued 1-cocyle for ξ^s. Then the flat τ-connection on ξ^s is defined by local sections ω^s_α in $q^*\Omega^1_X(\log D)$ which are d_τ flat for $d_\tau : q^*\Omega^1_X(\log D) \to q^*\Omega^2_X(\log D)$. So the cocyle condition reads $d_\tau \log g^s_{\alpha\beta} = \delta(\omega^s)_{\alpha\beta}$ where δ is the Cech differential. The Claim 5 implies then that $\mu_I^*(\omega^s_\alpha) \in (q_I^F)^*\Omega^1_{D_I}(\log(D_I \cap D^I))$, is τ-flat and 1 has $d_\tau \log \mu_I^*(g^s_{\alpha\beta}) = \delta\mu_I^*(\omega^s)_{\alpha\beta}$. So the class $(\xi_I^F)^s$ is defined by the Cech cocyle $(g^s_{\alpha\beta}, \mu_I^*\omega^s \oplus 0)$, with $\mu_I^*\omega^s \in B^1, 0 \in C^1$. □

We define a product

$$(\mathcal{K}_m\Omega^\infty_Q)_0 \times (\mathcal{K}_n\Omega^\infty_Q)_0 \xrightarrow{\cup} (\mathcal{K}_{m+n}\Omega^\infty_Q)_0 \tag{5}$$

by using the formulae defined in [5, Definition 2.1.1], that is

$$x \cup y = \begin{cases} \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ d_\tau \log x \wedge y \oplus d_\tau \log x \wedge y & x \in \mathcal{K}_m, y \in (B^n \oplus C^n)_{D_\tau} \\ 0 & \text{else.} \end{cases} \tag{6}$$

The product is well defined.

Definition 7. We define $c_n(q^*(E, \nabla, \Gamma)) \in \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty_Q))$ to be the image via the map $\mathbb{H}^n(Q, (\mathcal{K}_n\Omega^\infty_Q)_0) \to \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty_Q)$ of

$$\sum_{s_1 < s_2 \ldots < s_n} \xi^{s_1}(\nabla) \cup \cdots \cup \xi^{s_n}(\nabla).$$

Definition 8. On X we define the complex of sheaves

$$A_X(n) = A^n_X \to A^{n+1}_X \to \ldots$$

with

$$\begin{aligned} A^i_X &= B^i_X \oplus C^i_X \\ B^i_X &= \oplus_I (i_I)_*\Omega^i_{D_I}(\log(D^I \cap D_I)), \\ C^i_X &= \oplus_{I \neq \emptyset}(i_I)_*\Omega^{i-1}_X(\log D)|_{D_I}, \end{aligned}$$

where $C^i_X = 0$ for $i = n$. The differentials D_X are defined as follows: $(\oplus_I \beta_I, \oplus_I \gamma_I)$, where $\beta_I \in (i_I)_*\Omega^i_{D_I}(\log(D^I \cap D_I)), \gamma_I \in (i_I)_*\Omega^{i-1}_X(\log D)|_{D_I}$ is sent to

$$\begin{aligned} &\oplus_I d\beta_I \in (i_I)_*\Omega^{i+1}_{D_I}(\log(D^I \cap D_I)), \\ &\qquad \oplus_I d\gamma_I + (-1)^i(i_I^*\beta - \beta_I) \in (i_I)_*\Omega^i_X(\log D)|_{D_I}, \end{aligned}$$

where the differentials d_τ are the τ differentials in the various differential graded algebras $\Omega^\bullet_{D_I}(\log(D^I \cap D_I))$.

One has an injective morphism of complexes

$$\iota : \Omega_X^{\geq n} \to A_X^{\geq n} \tag{7}$$

sending $\alpha \in \Omega_X^i$ to $i_I^* \alpha \oplus 0$.

Proposition 9. *The morphism ι is a quasi-isomorphism. Furthermore, one has $Rq_* A(n) = A_X(n)$.*

Proof. We start with the second assertion: since μ_I is a closed embedding, one has $R(\mu_I)_* = (\mu_I)_*$ on coherent sheaves. Thus by the commutativity of the diagram (2), and the fact that $\mathcal{O}$ on the flag varieties is relatively acyclic, one has $Rq_*(R\mu_I)_*(q_I^F)^*\mathcal{E} = (i_I)_*\mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on D_I. This shows the second statement. We show the first assertion. We first show that the 0th cohomology sheaf of $A_X(n)$ is $(\Omega_X^n)_d$. The condition $D(\beta, \beta_I) = 0$ means $d\beta = d\beta_I = 0$ and $i_I^*\beta = \beta_I$. Thus $\beta \in \Omega_X^n$ and $d\beta = 0$. Assume now $i \geq n+1$. Then modulo $DA^{i-1}(n)$, $((\beta, \beta_I), \gamma_I)$ is equivalent to $((\beta, \beta_I + (-1)^{i-1} d\gamma_I), 0)$. So we are back to the computation as in the case $i = n$ and $\mathrm{Ker}(D)$ on $B^i \oplus 0$ is $\mathrm{Ker}(d)$ on Ω_X^i. On the other hand, by the same reason, $D(B^{i-1} \oplus C^{i-1}) = D(B^{i-1} \oplus 0)$, and $D(B^{i-1} \oplus 0) \cap (B^i \oplus 0) = d(\Omega_X^i)$. This finishes the proof. □

Proposition 10. *The map $q^* : AD^n(X)_\infty = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{\dim_X}) \to \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty)$ is injective. The classes $c_n((q^*(E, \nabla, \Gamma)) \in \mathbb{H}^n(Q, \mathcal{K}_n\Omega^\infty)$ in Definition 7 are of the shape $q^* c_n((E, \nabla, \Gamma))$ for uniquely defined classes $c_n((E, \nabla, \Gamma)) \in \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \to \Omega^{\dim_X})$.*

Proof. One has a commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
H^{n-1}(Q, \mathcal{K}_n) & \longrightarrow & \mathbb{H}^{n-1}(A(n)) & \longrightarrow & \mathbb{H}^n(\mathcal{K}_n\Omega_Q^\infty) & \longrightarrow & H^n(Q, \mathcal{K}_n) \\
\uparrow{\scriptstyle \mathrm{inj}} & & \uparrow{\scriptstyle =} & & \uparrow & & \uparrow{\scriptstyle \mathrm{inj}} \\
H^{n-1}(X, \mathcal{K}_n) & \longrightarrow & \mathbb{H}^{n-1}(A(n)_X) & \longrightarrow & \mathbb{H}^n(\mathcal{K}_n\Omega_X^\infty) & \longrightarrow & H^n(X, \mathcal{K}_n)
\end{array} \tag{8}$$

where $\mathcal{K}_n\Omega_X^\infty = \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \xrightarrow{d} \ldots \xrightarrow{d} \Omega_X^{\dim X}$. We write $H^i(Q, \mathcal{K}_j) = H^i(X, \mathcal{K}_j) \oplus$ rest, where the rest is divisible by the classes of powers of the $[\xi^s] \in H^1(Q, \mathcal{K}_1)$, with coefficients in some $H^a(X, \mathcal{K}_b)$. But $[\xi^s]$ comes by Lemma 6 from a class $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega_Q^\infty)_0)$. Consequently, the image of rest in $\mathbb{H}^i(A(n))$ dies. We conclude that one has an exact sequence $0 \to \mathbb{H}^n(\mathcal{K}_n\Omega_X^\infty) \to \mathbb{H}^n(\mathcal{K}_n\Omega_Q^\infty) \to \mathbb{H}^n(X, R^\bullet q_*\mathcal{K}_n/q_*\mathcal{K}_n)$. By the standard splitting principle for Chow groups, one has $H^n(Q, \mathcal{K}_n)/H^n(X, \mathcal{K}_n) = \mathbb{H}^n(X, R^\bullet q_*\mathcal{K}_n/q_*\mathcal{K}_n)$, and

$$\sum_{s_1 < s_2 \ldots < s_n} c_1(\xi^{s_1}) \cup \cdots \cup c_1(\xi^{s_n}) \in \mathrm{Im}(CH^n(X) \subset CH^n(Q)).$$

By Lemma 6, $\xi^s(\nabla) \in \mathbb{H}^1(Q, (\mathcal{K}_1\Omega^\infty)_0)$ maps to $c_1(\xi^s) \in H^1(Q, \mathcal{K}_1)$. Thus we conclude that $c_n(q^*(E, \nabla, \Gamma)) \in \mathrm{Im}(\mathbb{H}^n(\mathcal{K}_n\Omega^\infty_X) \subset \mathbb{H}^n(\mathcal{K}_n\Omega^\infty_Q)$. This finishes the proof. □

Theorem 11. *Let $X \supset U$ be a smooth (partial) compactification of a variety U defined over a characteristic 0 field, such that $D = \sum_j D_j = X \setminus U$ is a strict normal crossings divisor. Let (E, ∇) be a flat connection with logarithmic poles along D such that its residues Γ_j along D_j are all nilpotent. Then the classes $c_n((E, \nabla)) \in AD^n(X, D)$ lift to well defined classes $c_n((E, \nabla, \Gamma)) \in AD^n(X)$. They are functorial: if $f : Y \to X$ with Y smooth, such that $f^{-1}(D)$ is a normal crossings divisor, étale over its image $\subset D$, then $f^*c_n((E, \nabla, \Gamma)) = c_n(f^*(E, \nabla, \Gamma))$ in $AD^n(Y)$. If $D' \supset D$ is a normal crossings divisor and ∇' is the connection ∇, but considered with logarithmic poles along D', thus with trivial residues along the components of $D' \setminus D$, then $c_n((E, \nabla, \Gamma)) = c_n((E, \nabla', \Gamma'))$. The classes $c_n((E, \nabla, \Gamma))$ satisfy the Whitney product formula. In addition, $c_n((E, \nabla, \Gamma))$ lies in the subgroup $AD^n_\infty(X) = \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega^n_{\bar{X}} \xrightarrow{d} \Omega^{n+1}_X \to \ldots \xrightarrow{d} \Omega^{\dim(X)}_X) \subset AD^n(X)$ of classes mapping to 0 in $H^0(X, \Omega^{2n}_X)$. The restriction to $AD^n_\infty(D_I)$ of $c_n((E, \nabla, \Gamma))$ is $c_n((gr(F^\bullet_I), \nabla_I, \Gamma_I))$ where $(gr(F^\bullet_I), \nabla_I, \Gamma_I)$ is the canonical filtration (see Claim 4 and Definition 3).*

Proof. The construction is the Proposition 10. We discuss functoriality. If f is as in the theorem, then the filtrations $F^\bullet_I$ for (E, ∇) restrict to the filtration for $f^*(E, \nabla)$. Whitney product formula is proven exactly as in [12, 2.17, 2.18] and [8, Theorem 1.7], even if this is more cumbersome, as we have in addition to follow the whole tower of $F^\bullet_I$. Finally, the last property follows immediately from the definition of $\xi^s(\nabla)$ in Lemma 6. □

Theorem 12. *Assume given $k \subset \mathbb{C}$ and Γ is nilpotent. Then the classes $\hat{c}_n((E, \nabla)) \in H^{2n}((X \setminus D)_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$ defined in [12], come from well defined classes $\hat{c}_n((E, \nabla, \Gamma)) \in H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$. Furthermore $\hat{c}_n((E, \nabla, \Gamma))$ fulfill the same functoriality, additivity, restriction, and enlargement of ∇ properties as $c_n((E, \nabla, \Gamma)) \in AD^n_\infty(X)$.*

Proof. We just have to use the regulator map $AD^n(X) \to H^{2n-1}(X_{\mathrm{an}}, \mathbb{C}/\mathbb{Z}(n))$, which is an algebra homomorphism, and which defined in [8, Theorem 1.7]. Of course we can also follow the same construction directly in the analytic category. □

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